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On monoparametric families of orbits sufficient for integrability of planar potentials with linear or quadratic invariants

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Abstract. By combining Darboux's results on the direct construction of integrable systems with results from the inverse problem of dynamics, we prove that if a planar potential admits a monoparametric family of conic sections with constant focal distance or a family of confocal parabolas, it is integrable and the second invariant is quadratic in the moments, while the existence of a family of straight lines, either parallel or interesecting at one point also suffices for integrability, the second invariant being linear in the momenta.

1. Introduction

It is known—although never to our knowledge explicitly stated—that if a planar Hamiltonian of the form

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+V(x,y) \tag{1}$$

admits the monoparametric family of circular orbits

$$r^2=x^2+y^2=\text{constant} \tag{2}$$

then it is integrable. This conclusion may be drawn easily by combining results of the inverse problem of dynamics with Darboux's (1901) results on the direct construction of integrable potentials possessing a second integral of motion quadratic in the momenta. More specifically, it is known that all potentials of the form

$$V(r,\theta)=h(r)+r^{-2}g(\theta) \tag{3}$$

where r, θ are polar coordinates and h, g are arbitrary functions, are integrable, the second integral being

$$I=(xp_y-yp_x)^2+2g(\theta) \tag{4}$$

while at the same time the form (3) is the most general form for a potential which admits the family of circular orbits (2). This latter result was obtained by Broucke and Lass (1976) as an application of Joukovsky's theorem (Wittaker 1944, p 109) and by Bozis and Mertens (1985).

The aim of this paper is to determine other cases in which the existence of a monoparametric family of orbits

$$f(x,y)=\text{constant} \tag{5}$$

in a planar Hamiltonian system guarantees its integrability. In such cases, information on the geometry of the orbits in a system, which can be provided by observation, may reveal information about its integrability.

In the next two sections we briefly present some known results on the construction of integrable potentials by Darboux's method and on the inverse problem of dynamics respectively. Then, combining these results, we show that if one of the following monoparametric families of conic sections (modulo translations and/or rotations):

$$\frac{x^2}{A}+\frac{y^2}{A-c}=1 \tag{6}$$

or

$$r(1\pm\sin\theta)=A \tag{7}$$

is admitted by a planar Hamiltonian system, then the system is integrable, the second integral of motion being quadratic in the momenta.

In equations (6) and (7) A is the parameter, while c in (6) is any preassigned constant which does not vary within the family. The particular case of circular orbits (2) is included in (6) and corresponds to $c=0$.

In section 5 a degenerate case is examined and it is shown that if one of the monoparametric families of straight lines

$$y+cx=A \qquad x/y=A$$

is admitted by a potential $V(x, y)$ then the system is integrable. In this case the potential is one dimensional or central, respectively, and the second integral of motion is linear in the momenta and corresponds to a linear or angular momentum.

2. Potentials possessing integrals of motion quadratic in the momenta

The general form of the integrable planar potentials which possess a second integral quadratic in the momenta has been obtained by Darboux (1901), except for two degenerate cases, presented recently by Dorizzi *et al* (1983) and Ankiewicz and Pask (1983). Darboux's result is also presented in Whittaker (1944, p 331). We briefly present here the main conclusion.

In order for a planar potential $V(x, y)$ to possess a second integral quadratic in the momenta it is necessary and sufficient for V to be a solution of one of the following equations (modulo translations and/or rotations):

$$xy(V_{xx}-V_{yy})+(y^2-x^2+c)V_{xy}+3yV_x-3xV_y=0 \tag{8}$$

$$x(V_{xx}-V_{yy})+2yV_{xy}+3V_x=0 \tag{9}$$

$$V_{xy}=0 \tag{10}$$

where subscripts in V denote partial differentiation. In the case (8), if $c\neq 0$, V is of the form

$$V=(h(u)-g(v))/(u^2-v^2) \tag{11}$$

where

$$2u^2=x^2+y^2+c+[(x^2+y^2+c)^2-4cx^2]^{1/2} \tag{12a}$$

$$2v^2=x^2+y^2+c-[(x^2+y^2+c)^2-4cx^2]^{1/2} \tag{12b}$$

and the second integral is

$$I=(xp_y-yp_x)^2+cp_x^2+2(v^2h(u)-u^2g(v))/(u^2-v^2) \tag{13}$$

while for $c=0$, V is of the form (3), I being given by (4).

In the case (9) V is of the form

$$V=\frac{1}{r}(h(r+y)+g(r-y)) \tag{14}$$

with

$$I=p_x(yp_x-xp_y)+[(r+y)g(r-y)-(r-y)h(r+y)]/r \tag{15}$$

while the degenerate case (10) corresponds to

$$V=h(x)+g(y) \tag{16}$$

with

$$I=p_x^2+2h(x). \tag{17}$$

In all cases, h and g are arbitrary functions of their respective arguments.

3. Potentials which produce a given monoparametric family of orbits

It is known that the potentials $V(x, y)$ which admit the monoparametric family of orbits

$$f(x, y)=\text{constant}=A \tag{18}$$

may be determined as solutions of Szebehely's equation (Szebehely (1974), see also Puel (1984)), which may be written in the form

$$V_x+\gamma V_y+\frac{2\Gamma}{1+\gamma^2}(E-V)=0 \tag{19}$$

where γ and Γ depend only on the family (18) and are given by

$$\gamma=f_y/f_x \qquad \Gamma=\gamma\gamma_x-\gamma_y \tag{20}$$

where $E=E(f)$ is the constant energy along each orbit of family (18). In order to solve Szebehely's equation (19), one must define in advance the function $E(f)$, i.e. one must know how the energy varies within the family. In order to obtain a partial differential equation for the general form of the potentials $V(x, y)$ which admit the family of orbits (18), one may eliminate E from (19) as follows. Since E depends only on f, we have

$$E_x=E_ff_x \qquad E_y=E_ff_y$$

that is

$$E_y=\gamma E_x. \tag{21}$$

Solving equation (19) with respect to E and inserting it into (21), after some tedious but straightforward calculations we obtain the equation

$$(V_{xx}-V_{yy})-\varkappa V_{xy}+\lambda V_x+\mu V_y=0 \tag{22}$$

where $\varkappa$, λ, μ depend only on the family (18) and are given by the equations

$$\varkappa = (1-\gamma^2)/\gamma \tag{23a}$$

$$\lambda = (-\gamma\Gamma_x + \Gamma_y)/\gamma\Gamma \tag{23b}$$

$$\mu = \lambda\gamma + 3\Gamma/\gamma. \tag{23c}$$

Equation (22) was first obtained by Bozis (1984) and also by Bozis and Mertens (1985) as a special case of an equation for a potential producing a family of orbits on a given surface. The general solution of equation (22) for a particular f includes two arbitrary functions and corresponds to the most general form of a potential $V(x, y)$ compatible with the monoparametric family of orbits (18).

4. Families of orbits sufficient for integrability

Let again

$$f(x, y) = \text{constant} = A$$

be a monoparametric family of orbits. The general solution of equation (22) for this particular f corresponds to the most general form of the potential compatible with the above family. If at the same time equation (22) is identical to one of equations (8), (9) or (10), these potentials are integrable and the presence of this family of orbits in a planar potential suffices for its integrability. In the following, we will determine all the families of orbits for which the above remark is true.

Case (a1), $c \neq 0$. In order for equation (22) to be identical to equation (8), the following relations must hold:

$$\varkappa = (x^2 - y^2 - c)/xy \qquad \lambda = 3/x \qquad \mu = -3/y. \tag{24}$$

By taking into account equations (23), equations (24) become

$$\gamma = \tfrac{1}{2}\{-(x^2-y^2-c)/xy \pm [(x^2-y^2-c)^2/x^2y^2+4]^{1/2}\} \tag{25a}$$

$$\Gamma = -\gamma(\gamma y + x)/xy \tag{25b}$$

$$-\gamma\Gamma_x + \Gamma_y = 3\Gamma\gamma/x. \tag{25c}$$

If for γ given by equation (25a), equations (25b) and (25c) are true, then such a family of orbits does exist and the corresponding function f can be obtained as a solution of equation (20a), i.e.

$$f_y - \gamma f_x = 0. \tag{26}$$

Differentiating (25b) with respect to x and y, is easy to show that equation (25c) is valid, while it takes more elaborate algebra to show that (25b) is also true. In order to solve equation (26) for γ given by (25a), we perform the transformation to u, v given by equations (12). Then γ takes the forms

$$\gamma = u(c-v^2)^{1/2}/v(u^2-c)^{1/2}$$

or

$$\gamma = -v(u^2-c)^{1/2}/u(c-v^2)^{1/2}$$

and equation (26) becomes simply

$$f_u = 0 \qquad \text{or} \qquad f_v = 0$$

so the corresponding family is given by the equation

$$f = r^2 + c \pm [(r^2 + c)^2 - 4cx^2]^{1/2} = \text{constant} = 2A. \tag{27}$$

Equation (27) can also be written in the form

$$\frac{x^2}{A} + \frac{y^2}{A - c} = 1. \tag{28}$$

In (28), c is a pre-assigned constant which does not vary within the family while A is the parameter. Family (28) corresponds to conic sections with a constant focal distance $2|c|^{1/2}$.

Case (a2), $c = 0$. In this case the transformation (12) cannot be applied. For $c = 0$, the first equation of (24) yields the solutions

$$\gamma = y/x \qquad \text{or} \qquad \gamma = -x/y.$$

The second solution is unacceptable since it corresponds to $\Gamma = 0$ and in this case the procedure for deriving equation (22) cannot be applied. The degenerate case $\Gamma = 0$ will be treated separately in the next section. It can be easily checked that the first solution satisfies the two remaining equations (24). Equation (26) then yields the monoparametric family of circular orbits

$$x^2 + y^2 = \text{constant} = A$$

which may be included in (28) for $c = 0$.

Case (b). In this case we identify equation (22) with equation (9) and obtain the relations

$$\varkappa = -2y/x \qquad \lambda = 3/x \qquad \mu = 0. \tag{29}$$

The first of equations (29) yields

$$\gamma = (y \pm r)/x \tag{30}$$

and it can be shown by straightforward algebra that the other two equations are true for γ given by (30). The corresponding family of orbits may be determined by equation (26), which in this case, and in polar coordinates, has the form

$$\pm r f_r + \frac{1 \mp \sin\theta}{\cos\theta} f_\theta = 0$$

and yields the family of confocal parabolas

$$r(1 \pm \sin\theta) = A. \tag{31}$$

It is impossible to identify equation (22) with equation (10) in the degenerate case.

5. The degenerate case $\Gamma = 0$

Function Γ is proportional to the curvature of the orbits (Puel 1984) so this case corresponds to monoparametric families of straight lines.

In this case, the general form of a potential compatible with a monoparametric family of orbits with $\Gamma = 0$ is obtained by the general solution of Szebehely's equation (19), which now becomes

$$f_x V_x + f_y V_y = 0. \tag{32}$$

On the other hand, it is known (Hietarinta 1986, p 27) that the necessary and sufficient condition for a potential $V(x, y)$ to be integrable with a second integral of motion linear in the momenta is to be a solution of the equation

$$(ay + c) V_x + (-ax + b) V_y = 0 \tag{33}$$

where a, b and c are constants. We identify equation (32) with equation (33) and consider the following two cases.

Case (a). ($a = 0$). In this case, without loss of generality we may take $b = 1$ and obtain the equation

$$\gamma = f_y / f_x = 1/c$$

which gives $\Gamma = 0$ and corresponds to the parallel straight lines

$$y + cx = A. \tag{34}$$

In this case the potential is $V = V(cy - x)$ and the second invariant is $I = p_y + cp_x$.

Case (b) ($a \neq 0$). We may perform a translation and take $b = c = 0$, while a can be put equal to 1. Identifying (33) with (32), we obtain

$$\gamma = f_y / f_x = -x/y$$

which also corresponds to $\Gamma = 0$ and yields the family of straight lines

$$x/y = A. \tag{35}$$

The corresponding potential is central and I is the angular momentum.

6. Conclusions

It was shown in this paper that information about the geometry of the orbits in a planar Hamiltonian system may reveal information about its integrability. The existence of a monoparametric family of orbits in such a system may suffice for the system to be integrable. More specifically, if a planar potential $V(x, y)$ admits one of the following families of conic sections:

$$\frac{x^2}{A} + \frac{y^2}{A - c} = 1 \qquad r(1 \pm \sin\theta) = A$$

or one of the families of straight lines

$$y + cx = A \qquad x/y = A$$

where c is a particular preassigned constant and A is the parameter, then the system is integrable, the second integral being quadratic in the momenta in the first case and linear in the second one.

This result was obtained by identifying Bozis–Mertens or Szebehely's equations (22) or (19) with equations (8) and (9) or (33), which correspond to sufficient conditions for the existence of a second integral of motion, quadratic or linear in the momenta respectively.

The same method may be applied for integrability with a second integral of degree higher than 2 in the momenta, but the results on the construction of such integrable potentials are too incomplete (e.g. Hietarinta 1986, p 36 ff) for a definite conclusion to be drawn.

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